THE NAVIER-STOKES EQUATIONS IN TWO DIFFERENT FORMULATIONS WITH MODERATE AND HIGH REYNOLDS NUMBERS

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Abstract. The goal of this work is to present results for 2D viscous incompressible flows governed by the Navier-Stokes equations. Two different formulations will be used: The Stream Function-vorticity and the Velocity-vorticity formulation. To show that the schemes we are using are working for moderate and high Reynolds numbers, we are going to report results for the very well known un-regularized driven cavity problem, with Reynolds numbers in the range of $3200 \leq Re \leq 50000$.

1 INTRODUCTION

We are going to work with the Navier-Stokes in two different formulations: The Stream Function-vorticity and the Velocity-vorticity formulation. The problem we are going to solve is the well known un-regularized driven cavity problem, with Reynolds numbers in the range of $3200 \leq Re \leq 50000$. Results, in both formulations, are obtained using a simple numerical scheme based on a fixed point iterative process (see [1]), applied to a nonlinear elliptic system resulting after time discretization. The scheme has shown to be robust enough to handle such Reynolds numbers, from moderate to high, which is not an easy task to deal with (see [2] and [3]). As the Reynolds number increases the mesh has to be refined and a smaller time step has to be used, numerically, by stability matters and physically, to capture the fast dynamics of the flow, as pointed out in [4], although, with the Velocity-vorticity formulation (see [5] and [6]), a finer mesh has to be used. So, because of this, computing time is in general very large with this numerical scheme and for both formulations, so we seek to reduce this time by, instead of working only with the matrix A, resulting from the discretization of the Laplacian term, using both matrixes A and B, the second one resulting from the discretization of the advective term. For the Stream Function-vorticity formulation and moderate and high Reynolds numbers, this second scheme has been faster than the fixed point iterative method (see [7], [8]). For the Velocity-vorticity formulation we are just showing results using the fixed point iterative method, and we are still looking forward to modify this scheme by using here too both matrixes A and B and in this way being able to reduce computing time when using this formulation.

2 Mathematical Models

Let $\Omega \subset R^N$ $(N = 2, 3)$ the region of a nonsteady, viscous, incompressible flow, and Γ its boundary.

$$\begin{cases} \frac{\partial u}{\partial t} - \nabla^2 \mathbf{u} + \nabla p + (\mathbf{u} \cdot \nabla)\mathbf{u} & = f, \qquad (a) \\ \nabla \cdot \mathbf{u} & = 0 \qquad (b) \end{cases} \tag{1}$$

These are the Navier-Stokes equations in primitive variables and this system has to be supplemented with appropiate boundary and initial conditions.

2.1 The Stream function-vorticity Formulation

We will restrict ourselves to a bidimensional region Ω. Taking the curl in both sides of the equation $(1a)$ and taking into account that

$$\begin{cases} u_1 = \frac{\partial \psi}{\partial y}, \quad u_2 = -\frac{\partial \psi}{\partial x}, \end{cases} \tag{2}$$

which follows from $(1b)$, with ψ the stream function and u_1, u_2, the two components of the velocity. So we get:

$$\begin{cases} \nabla^2 \psi & = -\omega \qquad (a) \\ \frac{\partial \omega}{\partial t} - \frac{1}{Re}\nabla^2 \omega + \mathbf{u} \cdot \nabla \omega & = f_\omega \qquad (b) \end{cases} \tag{3}$$

where ω is the vorticity $(\omega = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y})$. These are the Navier-Stokes equations in the Stream function-vorticity formulation. The inconmpressibility condition $(1b)$, by (2) is automatically satisfied, and the pressure does not appear any more, which is a significant advantage with respect to the primitive variable formulation.

2.2 Velocity-Vorticity Formulation

Taking the curl in

$$\omega = -\nabla \times \boldsymbol{u} \tag{4}$$

and using the identity $\nabla \times \nabla \times \mathbf{a} = -\nabla^2 \mathbf{a} + \nabla(\nabla \cdot \mathbf{a})$ and $(1b)$, a velocity Poisson equation results:

$$\nabla^2 \mathbf{u} = -\nabla \times \boldsymbol{\omega}. \tag{5}$$

Then, two Poisson equations for the velocity components are obtained, which together with the equation for the vorticity gives us

$$\left\{ \begin{array}{ll} \frac{\partial u_1}{\partial t} + \nabla^2 u_1 = -\frac{\partial \omega}{\partial y} & (a) \\ \frac{\partial u_2}{\partial t} + \nabla^2 u_2 = \frac{\partial \omega}{\partial x}\,. & (b) \\ \frac{\partial \omega}{\partial t} - \frac{1}{Re}\nabla^2 \omega + \mathbf{u}\cdot\nabla\omega = f_\omega & (c) \end{array} \right. \tag{6}$$

These are de Navier-Stokes equations in the Velocity-vorticity Formulation.

3 The Numerical Schemes.

For the time derivative appearing in the vorticity equation in both schemes, the following second order approximation is used:

$$\frac{\partial f}{\partial t}(\mathbf{x}, (n+1)\Delta t) = \frac{3f^{n+1} - 4f^n + f^{n-1}}{2\Delta t} \tag{7}$$

where $\mathbf{x}\,\epsilon\,\Omega$, $n \geq 1$, Δt denotes the time step, and $f^r \approx f(\mathbf{x}, r\Delta t)$, assuming f is smooth enough.

3.1 The Stream function-Vorticity formulation

The following nonlinear elliptic system has to be solved at each time level:

$$\left\{ \begin{array}{ll} \nabla^2 \psi = -\omega, \quad \psi|_\Gamma = \psi_{bc}; & (a) \\ \alpha\omega - \nu\nabla^2\omega + \mathbf{u}\cdot\nabla\omega = f_\omega, \quad \omega|_\Gamma = \omega_{bc}\,, & (b) \end{array} \right. \tag{8}$$

where $\alpha = \frac{3}{2\Delta t}$, $\nu = \frac{1}{Re}$ and $f_\omega = \frac{4\omega^n - \omega^{n-1}}{2\Delta t}$. To obtain (ψ^1, ω^1), the first subinterval is divided into M subintervals, and a first order scheme, such as Euler, is applied to each of the M subintervals.

Let R_ω be defined by:

$$R_\omega(\omega,\psi) \equiv \alpha\omega - \nu\nabla^2\omega + \mathbf{u}\cdot\nabla\omega - f_\omega \, . \tag{9}$$

So, system (8) is equivalent to:

$$\left\{\begin{array}{lcl} \nabla^2\psi & = -\omega & in\ \Omega\,, \quad \psi = \psi_{bc} \quad on\ \ \Gamma \\ R_\omega(\omega,\psi) & = \ \ 0 & in\ \Omega \quad \omega|_\Gamma = \omega_{bc} \end{array}\right. \tag{10}$$

This system is solved, at time level (n+1), by the following fixed point iterative process:

Given $\omega^{n,0} = \omega^n$, $\psi^{n,0} = \psi^n$ *solve until convergence in* ω *and* ψ

$$\left\{\begin{array}{l} \nabla^2\psi^{n,m+1} = -\omega^{n,m} \quad in\ \Omega, \\ \psi^{n,m+1} = \psi_{bc}^{n,m+1}\ on\ \Gamma \\ (\alpha I - \nu\nabla^2)\omega^{n,m+1} = (\alpha I - \nu\nabla^2)\omega^{n,m} - \rho_\omega R_\omega(\omega^m,\psi^{n,m+1})\ in\ \Omega, \\ \omega^{n,m+1} \ = \omega_{bc}^{n,m+1}\ on\ \Gamma,\ \rho_\omega > 0. \end{array}\right. \tag{11}$$

and then, take $(\omega^{n+1},\psi^{n+1}) = (\omega^{n,m+1},\psi^{n,m+1})$.

In order to reduce computing time, we worked on solving system by the following method at each time step:

$$\left\{\begin{array}{ll} \nabla^2\psi^{n+1} = -\omega^n, \quad \psi^{n+1}|_\Gamma = \psi_{bc}^{n+1}; & (a) \\ (\alpha I - \nu A)\omega^{n+1} + B\omega^{n+1} = f_\omega, \quad \omega^{n+1}|_\Gamma = \omega_{bc}^{n+1}\,, & (b) \end{array}\right. \tag{12}$$

Here, A and B are the matrixes asociated with the discretization of the difussive (Laplacian) and the advective term, respectively. The linear system of equations resulting, is solved using Gauss-Seidel.

3.2 The Velocity-Vorticity Formulation

For the time derivatives appearing in the vorticity equation (7) is used, and the following fully implicit time discretization system is obtained, in Ω,

$$\left\{\begin{array}{ll} \frac{\partial u_1}{\partial t} + \nabla^2 u_1 & = -\frac{\partial\omega}{\partial y} \\ \frac{\partial u_2}{\partial t} + \nabla^2 u_2 & = \frac{\partial\omega}{\partial x}, \quad \mathbf{u}^{n+1}|_\Gamma = \mathbf{u}_{bc} \\ R_\omega(\omega,\mathbf{u}) = \mathbf{0}, & \omega|_{\mathbf{\Gamma}} = \omega_{\mathbf{bc}} \end{array}\right. \tag{13}$$

where

$$R_\omega(\omega, \mathbf{u}) \equiv \alpha\omega - \nu\nabla^2\omega + \mathbf{u}\cdot\nabla\omega - \mathbf{f}_\omega \ , \tag{14}$$

and using again the fixed point iterative method, we have:

Given $\omega^{n,0} = \omega^n$, $u_1^{n,0} = u_1^n$, $u_2^{n,0} = u_2^n$ *solve until convergence on* ω, u_1 and u_2

$$\begin{cases} \frac{\partial u_1^{n,m+1}}{\partial t} + \nabla^2 u_1^{n,m+1} = -\frac{\partial \omega^{n,m}}{\partial y} \\ \frac{\partial u_2^{n,m+1}}{\partial t} + \nabla^2 u_2^{n,m+1} = \frac{\partial \omega^{n,m}}{\partial x}, \qquad \mathbf{u}^{n,m+1}|_\Gamma = \mathbf{u}_{bc}^{n,m+1} \\ (\alpha I - \nu\nabla^2)\omega^{n,m+1} = (\alpha I - \Delta)\omega^m - \rho_\omega R_\omega(\omega^{n,m}, \mathbf{u}^{n,m+1}), \\ \quad \rho_\omega > 0, \ \omega^{n,m+1}|_\Gamma = \omega_{bc}^{n,m}. \end{cases} \tag{15}$$

and then, take $(\omega^{n+1}, u_1^{n+1}, u_2^{n+1}) = (\omega^{n,m+1}, u_1^{n,m+1}, u_2^{n,m+1})$.

4 Numerical experiments

The numerical experiments take place in rectangular domains $\Omega = (0,a)\times(0,b)$, $a, b > 0$, in connection with the lid-driven cavity problem (in our case, $a = 1$ and $b = 1$). The boundary condition of $\mathbf{u}$ is given by $\mathbf{u} = (1,0)$ at the moving boundary $y = b$ and $\mathbf{u} = (0,0)$ elsewhere.

4.1 Stream function-vorticity formulation

A translation of the boundary condition in terms of the velocity primitive variable $\mathbf{u}$ to the $\psi - \omega$ variables has to be performed. Following [9], $\psi = 0$ is chosen on Γ, and by Taylor expansion of (8a) on the boundary, with h_x and h_y the space steps, one obtains:

$$\begin{cases} \omega(0,y,t) = -\frac{1}{2h_x^2}[8\psi(h_x,y,t) - \psi(2h_x,y,t)] + O(h_x^2) \\ \omega(a,y,t) = -\frac{1}{2h_x^2}[8\psi(a-h_x,y,t) - \psi(a-2h_x,y,t)] + O(h_x^2) \\ \omega(x,0,t) = -\frac{1}{2h_y^2}[8\psi(x,h_y,t) - \psi(x,2h_y,t)] + O(h_y^2) \\ \omega(x,b,t) = -\frac{1}{2h_y^2}[8\psi(x,b-h_y,t) - \psi(x,b-2h_y,t)] - \frac{3}{h_y} + O(h_y^2). \end{cases} \tag{16}$$

With both schemes mentioned above (using just matrix A with the fixed point iterative method, and using A and B) results agree very well, so we present the graphs obtained with the second scheme. The advantage of the second scheme is that the computing time requiered is almost half of the time needed for the fixed point iterative process.

In Figure 1 we show the streamlines and isovorticity conturs for $Re = 25000$ with $h = h_x = h_y = 1/512$. In Figure 2 we show results for $Re = 31000$ and the same value of h. In figure 3, we show results for $Re = 50000$, with $h = h_x = h_y = 1/1024$. For these Reynolds numbers, since there is no steady state, results are shown for $T_f inal = 5$.

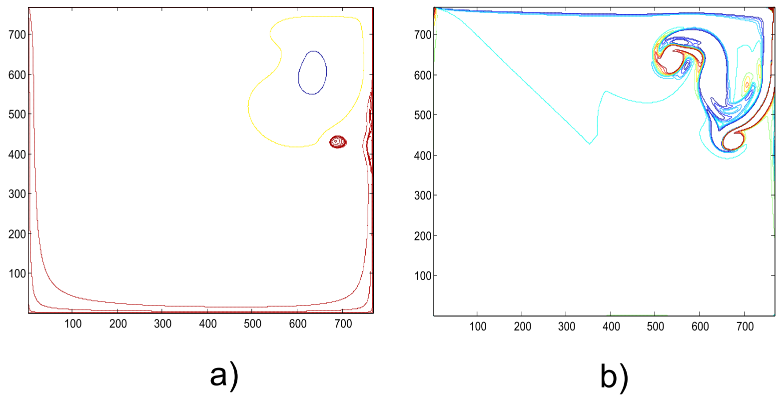


Figure 1: Streamlines (left) and isovorticity contours(right)for $Re = 25000, h = h_x = h_y = 1/728, dt = .00025, T_{final} = 5$

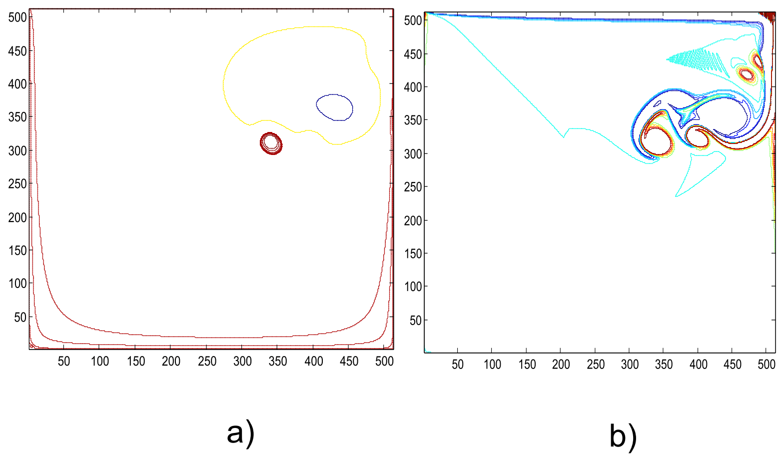


Figure 2: Streamlines (left) and isovorticity contours (right) for $Re = 31000, h = h_x = h_y = 1/512, dt = .00025, T_{final} = 5$

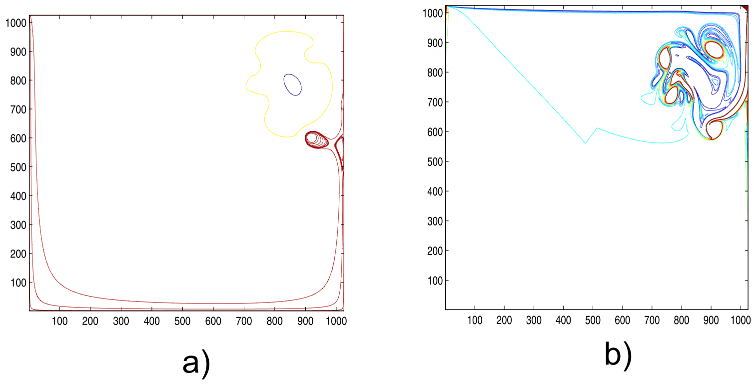


Figure 3: Streamlines (left) and isovorticity contours (right) for $Re = 50000, h = h_x = h_y = 1/1024, dt = .00025, T_{final} = 5$

4.2 Velocity-Vorticity formulation

The boundary conditions are given by:

$$\begin{cases} \mathrm{u}_1 = 0,\ u_2 = 0,\ \omega = \frac{\partial u_2}{\partial x} \text{ on } \Gamma_x = 0 \\ \mathrm{u}_1 = 0,\ u_2 = 0,\ \omega = \frac{\partial u_2}{\partial x} \text{ on } \Gamma_x = 1 \\ \mathrm{u}_1 = 0,\ u_2 = 0,\ \omega = -\frac{\partial u_1}{\partial y} \text{ on } \Gamma_y = 0 \\ \mathrm{u}_1 = 1,\ u_2 = 0,\ \omega = \frac{\partial u_1}{\partial x} \text{ on } \Gamma_y = 1 \end{cases} \tag{17}$$

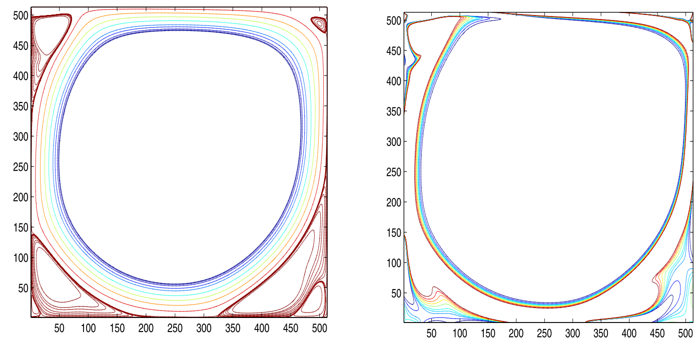
The initial conditions are given by $\mathbf{u}(\mathbf{x}, \mathbf{0}) = (\mathbf{0}, \mathbf{0})$ and $\omega(\mathbf{x}, \mathbf{0}) = \mathbf{0}$

As remarked in [5], not all the results could be obtained with second order discretization. A fourth order discretization was required . This is accomplished by using the fourth order option of Fishpack ([10]) which was used in this work, for solving the elliptic problems arising.

In Figure 4 we show the streamlines and isovorticity contours for $Re = 3200, h = h_x = h_y = 1/512, T_{final} = 50$.

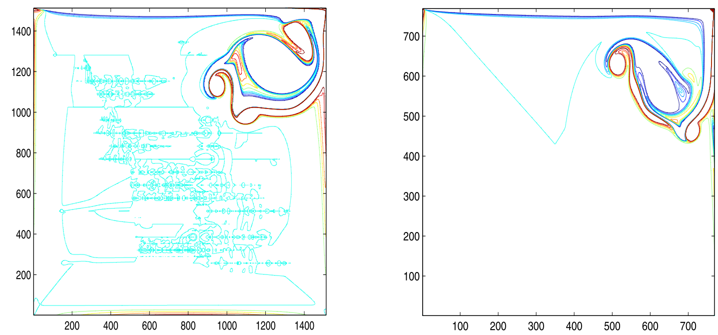
Finally, in Figure 5, we show just the vorticity contours For $Re = 20000$ with a) $h = h_x = h_y = 1/1512, T_{final} = 5$, obtained using the Velocity-vorticity formulation, and b) with the Stream Function-vorticity Formulation with $h = h_x = h_y = 1/768, T_{final} = 5$.

As can be seen, with the Stream Function-vorticity formulation we are using an h half of the one used for the Stream Function-vorticity formulation and we think that the results obtained with this formulation are much reliable; even more, the computing time used to obtain the results with the Velocity-vorticity formulation was much more greater than the one for the Stream Function-vorticity formulation. We think that there are still some numerical problems with this formulation for high Reynolds numbers.



a) b)

Figure 4: Streamlines (left) and isovorticity contours (right) for $Re = 3200, h = h_x = h_y = 1/512, dt = .0001, T_{final} = 50$



(a) (b)

Figure 5: Isovorticity contours for $Re = 20000$, a) Velocity-vorticity formulation with $h = h_x = h_y = 1/1512, dt = .0001, T_{final} = 5$, b) Stream Function-vorticity formulation with $h = h_x = h_y = 1/768, dt = .0001, T_{final} = 5$.

5 Conclusions

We are presenting efficient numerical methods for solving the Navier-Stokes equations in the Stream funtion-vorticity and the Velocity-vorticity formulations.

Results agree very well with those reported in the bibliography (see [2], [3], [4]), but with the Stream function-vorticity formulation and working with matrixes A and B resulting from the discretization of the Laplacian and the advective term respectively, we were able to reduce computing time to almost one half with respect to the Fixed point iterative method, especially for high Reynolds numbers, for which the computing time increases a lot. In this case, as we have said, smaller values of h have to be used; numerically for stability and physically to capture the fast dynamics of the flow. We are still looking forward to reduce computing time. Previous results have already been obtained for $Re = 75000$ using the Stream function-vorticity formulation; they will be reported in a future work.

For the Velocity-vorticity formulation, as mentioned, we are just showing results using the fixed point iterative method, and we are still looking forward to modify this scheme by using here too both matrixes A and B and in this way being able to reduce computing time when using this formulation. This is the reason we are just showing results for Re up to 20000 and we did not go further.

In conclusion, the numerical procedure applied to the Stream Function-vorticity formulation is not as good for the Velocity-vorticity formulation, however, the way it behaves, through the discretization parameters, and the order of discretization, gives us another point of view of the behavior of flows under different numerical methods and different formulations, as pointed out in [5]. The difficulty of the Velocity-vorticity formulation is reinforced through some works such as [11], who, with very different methods reported driven cavity flows for moderate Reynolds numbers, lower than ours.

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